LIFE AND WELL-BEING | SCIENCE

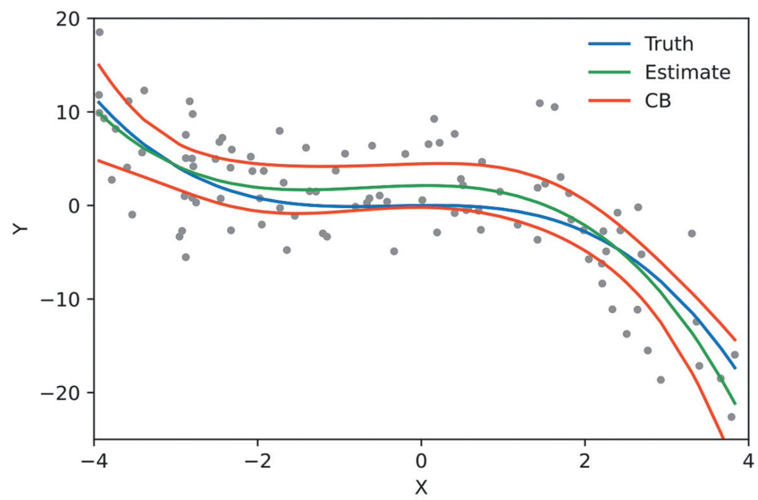
Linear algebra: the queen of applied mathematics

ALEXANDER FARRUGIA

"Mathematics is the art of reducing any problem to linear algebra." This is a quote often attributed to William Stein, a former mathematics professor at the University of Washington, now the lead developer of SageMath. Indeed, for many areas of science, engineering, technology and mathematics, reducing a problem to linear algebra is often the most straightforward and sure way of solving it efficiently.

Perhaps one of the most recent innovations is the arrival of large language models (LLMs) like ChatGPT, whose underlying mechanisms lie in linear algebra. Text is converted into a vector in an inner product space, to which linear algebra techniques are then applied to capture meaning, obtain semantic relationships and generate text through a 'transformer'.

Other AI systems like recommendation systems, used to predict user preferences, also rely on linear algebra. Additionally, linear algebra plays a key role in robotics to calculate robot joint positions and orientations, as well as facilitate image transformations and feature detection, which are important applications of computer vision.



Linear algebra is at the heart of least squares polynomial regression, a statistical technique widely used in several fields of study. PHOTO: WIKIPEDIA (SKBKEKAS, CC BY 3.0.)

In modern portfolio theory (MPT), which is key to finance, portfolio selection is formulated as a quadratic optimisation problem, aiming to maximise expected return for a given level of risk. In financial risk management, linear algebra provides tools for quantifying and decomposing risk from matrix representations of portfolio data. Eigenspaces are particularly useful in identifying the main constituents of risk through principal component analysis (PCA), an important statistical technique used in many other fields besides risk management.

The optimisation of the allocation of resources in economic systems is also carried out using linear algebra through matrix representations of variables like production and consumption.

In civil engineering, eigenvalue analysis helps identify a structure's natural vibrational modes, which is vital for preventing resonance failures. In electrical engineering, linear algebra techniques like Gaussian elimination are applied in circuit analysis to determine unknown currents and voltages.

Linear algebra is likewise useful in signal processing, where matrices represent filters for noise reduction. In control systems, linear state-space models describe dynamic behaviour, and eigenspaces are essential in the analysis of system stability and in the design of feedback controllers.

Eigenspaces are also fundamental in quantum mechanics, where they describe particle behaviour and energy states, as well as in algebraic and spectral graph theory, subfields of graph theory that reveal structures of networks algebraically.

Clearly linear algebra is an indispensable part of applied mathematics; its wide array of applications is hard to beat.

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CURIOSITIES

History of linear algebra

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Although linear algebra is regarded as a relatively modern mathematics topic, its ideas have been around for a long time.

The first instance of the solution of a system of linear equations, using what is nowadays called Gaussian elimination, can be found in the eighth chapter of the text *Nine Chapters on the Mathematical Art*, written by Chinese scholars around 3,000 years ago. In stark contrast, Gaussian elimination was introduced to the western world by Sir Isaac Newton in 1669.

The theory of determinants was, likewise, introduced to humankind by the eastern world prior to the western one. This time, Japan takes the honours; Seki Takakazu, known as the 'Japanese Newton', is credited to be the first who described determinants in 1683, a few years before Gottfried Wilhelm Leibniz's first work on the subject.

Perhaps surprisingly, matrices were envisioned after determinants. It was James Joseph Sylvester who coined the word 'matrix' – meaning 'womb' – in 1848. A few years earlier, Hermann Grassmann, considered the father of modern linear algebra, conceived what nowadays we call vector spaces and linear independence.

At around the same time, Sir William Rowan Hamilton invented the quaternions, and with them, the terms 'vector' – from the Latin word *vehere*, meaning 'to carry' – and 'scalar', used to scale vectors.

Later, in 1856, Arthur Cayley, who collaborated a lot with Sylvester, introduced matrix multiplication and matrix inversion, in the process linking matrices with determinants.

PHOTO OF THE WEEK

A photo of the author, split into its three colour constituents (red, green and blue) using linear algebra. These three images were produced by representing the original picture as a matrix and multiplying the matrix by an appropriate vector. Summing up the matrix representations of each of these three images yields the original picture.

SOUND BITES

- In the recent paper by Alexander Farrugia 'Recovering the characteristic polynomial of a graph from entries of the adjugate matrix', published in the *Electronic Journal of Linear Algebra*, 38 (2022), 697-711, the author links several entries of the adjugate matrix of a network to its characteristic polynomial. This has applications in the polynomial reconstruction problem in spectral graph theory, which asks if the characteristic polynomial of a graph can always be obtained from the characteristic polynomials of its vertex-deleted subgraphs. This problem, in turn, is related to the graph reconstruction conjecture of Kelly and Ulam.

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DID YOU KNOW?

- Linear algebra is the study of vectors, matrices, vector spaces and linear transformations. It emerged from the need to solve systems of linear equations, but has evolved into a discipline leading to other mathematical areas of study like functional analysis and modern geometry.
- The International Linear Algebra Society (ILAS) is an organisation dedicated to scientists, professionals and educators interested in linear algebra and its applications. Among other things, it publishes a journal, the *Electronic Journal of Linear Algebra*, organises conferences and awards prizes for distinguished research in linear algebra.

For more trivia, see: www.um.edu.mt/think.

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